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Some Results Concerning the Transition from the *L*- to the *P*-Property for Pairs of Finite Matrices*

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1. INTRODUCTION

Two $n \times n$ matrices A, B with elements in a field F which contains their characteristic roots $\{\alpha_i\}$, $\{\beta_i\}$ are said to have the L-property (a concept introduced by Kac) if all matrices $\lambda A + \mu B$ $(\lambda, \mu \in F)$ have $\lambda\alpha_i + \mu\beta_i$ as characteristic roots for a fixed ordering independent of λ, μ. They have been studied previously by Motzkin and Taussky [2].

If for all polynomials $p(x, y)$ in the noncommuting variables x, y the polynomial $p(A, B)$ has as characteristic roots $p(\alpha_i, \beta_i)$ then A, B are said to have the P-property (McCoy, see [2]). An equivalent property is that A, B can be transformed to triangular form simultaneously by a similarity.

Recently this author raised the question whether a group of matrices in which every pair has the L-property is similar to a group of triangular matrices. This was confirmed by Wales and Zassenhaus [4]. They use group theoretic methods. Here, a matrix treatment is used to study conditions on certain pairs of words in A, B which are sufficient to ensure that the additive commutator $AB - BA$ is nilpotent. In particular AB, BA are assumed to have the L-property. The multiplicative commutator will then be unipotent. If the L-property is further assumed for every pair of commutators then the commutators generate a unipotent normal subgroup of the group generated by A, B. Such a group is known to be equivalent to an upper triangular group [1], [5]. Both in [4] and here it is not assumed that the pairing of the roots α_i, β_i leads to an analogous pairing for the words used, but this property is deduced.

In 3, a number of facts about this problem are derived, while in 2, some

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general theorems concerning pairs A, B with the L-property are given. Some of these had been found previously in [2] and some additional ones are derived here. In 4, the problem is solved for some special cases.

The proofs are of somehow computational nature. While a more abstract treatment is desirable, the computational treatment gives much insight into the structure of the matrices.

2. Pairs of Matrices with the L-Property

In what follows we assume that A, B is a pair of $n \times n$ matrices with the L-property and that the pairing of the characteristic roots is α_i, β_i. This pairing is invariant under simultaneous similarity transformation of A and B. It will be assumed that one of the matrices, say A, is similar to a diagonal matrix. We may then assume that it is in fact the diagonal $(\alpha_1, \ldots, \alpha_n)$. In this case the pairing of the roots is explicitly displayed. For the following theorem holds.

Theorem 2.1 [Motzkin and Taussky [2]]. *Let A and B be as described above and arrange the characteristic roots in sets of equal ones such that α_i has multiplicity m_i. Partition the matrix B conformally with the m_i-dimensional scalar matrices in A. Then the $m_i \times m_i$ diagonal block of B contains the characteristic roots paired with α_i. Further*

$$\sum_{i \neq k} b_{ik} b_{ki} = 0 \tag{1}$$

when the summation is over all i, k outside of every diagonal block. If in particular all $m_i = 1$ then the summation is over all $i \neq k$.

Theorem 2.2 [2], [4]. *If A is nonsingular then $A^{-1}B$ has as its characteristic roots $\alpha_i^{-1}\beta_i$, $i = 1, \ldots, n$.*

Unless explicitly stated otherwise the α_i are assumed different from now on.

Theorem 2.3. *The following relations hold*:

$$\sum C_{\ell, i_1, \ldots, i_{r-1}} \Delta_{\ell, i_1, \ldots, i_{r-1}} = 0, \qquad \ell = 1, \ldots, n. \tag{2_r}$$

Here $1 < r \leqslant n$; $i_1 < i_2 < \cdots < i_{r-1}$ are indices different from ℓ, chosen from $1, 2, \ldots, n$; $C_{\ell, i_1, \ldots, i_{r-1}} = \prod (\alpha_\ell - \alpha_i)$, $i \neq \ell, i_1, \ldots, i_{r-1}$ and $\Delta_{\ell, i_1, \ldots, i_{r-1}}$ is the $r \times r$ principal minor formed from the indices $i_1, \ldots, i_{r-1}$, ℓ, but with the diagonal $\{b_{ii}\}$ replaced by $\{b_{ii} - b_{\ell\ell}\}$, hence containing a zero element.

Proof. In virtue of the L-property the determinant

$$|\lambda A + B - (\lambda\alpha_\ell - \beta_\ell)I|$$

vanishes identically. The coefficient of λ^{n-r} in this polynomial is the expression (2_r). Here we also use the fact that $b_{ii} = \beta_i$. For $r = 2$ and, e.g., $\ell = 1$ we obtain:

$$\sum_{i_1 \neq 1} \prod_{i \neq 1, i_1} (\alpha_1 - \alpha_i) \begin{vmatrix} 0 & b_{1i_1} \\ b_{i_1 1} & b_{i_1 i_1} - b_{11} \end{vmatrix} = \sum_{i_1 \neq 1} \prod_{i \neq 1, i_1} (\alpha_1 - \alpha_i)\, b_{1i_1} b_{i_1 1} = 0.$$

For $r = 3$ and $\ell = 1$ we obtain

$$\sum_{\substack{i_1 < i_2 \\ i_1, i_2 \neq 1}} \prod_{i \neq 1, i_1, i_2} (\alpha_1 - \alpha_i) \begin{vmatrix} 0 & b_{1i_1} & b_{1i_2} \\ b_{i_1 1} & b_{i_1 i_1} - b_{11} & b_{i_1 i_2} \\ b_{i_2 1} & b_{i_2 i_1} & b_{i_2 i_2} - b_{11} \end{vmatrix} = 0.$$

For $r = n - 1$ and $\ell = 1$

$$\sum_{\substack{i_1 < i_2 \cdots i_{n-2} \\ i_1 \neq 1}} (\alpha_1 - \alpha_i)\, \Delta_{1, i_1, \ldots, i_{n-2}} = 0, \qquad i \neq 1, i_1, \ldots, i_{n-2}.$$

For $r = n$, $\ell = 1$

$$\begin{vmatrix} 0 & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} - b_{11} & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{n1} & & \cdots & b_{nn} - b_{11} \end{vmatrix} = 0.$$

3. Some Theorems Concerning Pairs of Words in A, B Which Also Have the L-Property

We assume A, B as in 2.3 and from now on nonsingular.

Theorem 3.1. *Let A^{-1}, B have the L-property as well. Then the characteristic root $1/\alpha_i$ is paired with β_i, $i = 1, 2, \ldots, n$, and AB has the characteristic roots $\alpha_i\beta_i$.*

Proof. The first statement follows by applying Theorem 2.1 to A^{-1}, B instead of A, B. The second statement follows then from Theorem 2.2.

Theorem 3.2. *Let the following pairs have the L-property:*

$$A, A^r B, \qquad r = 0, \pm 1, \pm 2, \ldots .$$

Then

$$b_{ik}b_{ki} = 0, \qquad i \neq k, \qquad i, k = 1,..., n.$$

Remark. An immediate consequence of 3.2 is: if A, B^2 is also a pair with the L-property, then the pairing of their roots is α_i, β_i^2. For, by 2.1 the root corresponding to α_i is $\beta_i^2 + \sum_{r \neq i} b_{ir}b_{ri} = \beta_i^2$.

Proof of 3.2. We shall use Theorem 2.3 for $r = 2$ for the matrix pairs A, A^rB, $r = 1, 2,...$. By Theorem 2.2 the characteristic roots of A^rB are $\alpha_i^r\beta_i$. Here we also use the following fact: since A is a diagonal of different elements, the pairing of the characteristic roots in the pair A, A^rB is still α_i, $\alpha_i^r\beta_i$, since the latter elements form the diagonal of A^rB. Also the ordering is as in B. The other elements of A^rB are $b_{ik}\alpha_i^r$. When we then apply Theorem 2.3, $r = 2$, $l = 1$ to the pairs A, A^rB, $r = 0, 1, 2,...$, the coefficients c_{1,i_1} are not altered, but $b_{1k}b_{k1}$ is replaced by $b_{1k}b_{k1}\alpha_1^r\alpha_k^r$. In this way we obtain a system of linear equations

$$\sum_{k=2}^{n} \alpha_1^r\alpha_k^r \prod_{i \neq 1,k} (\alpha_1 - \alpha_i) \cdot b_{1k}b_{k1} = 0, \qquad i = 0,..., n-2$$

for the $n - 1$ quantities $\prod_{i \neq 1,k} (\alpha_1 - \alpha_i)\, b_{1k}b_{k1}$. The matrix of this system is a van der Monde for the elements α_2,..., α_n (the factor α_1 being ignored). Since these elements are different, the van der Monde does not vanish. In consequence

$$b_{1k}b_{k1} = 0, \qquad k \neq 1$$

and likewise

$$b_{ik}b_{ki} = 0, \qquad i \neq k.$$

In consequence of 3.2 we may replace the determinants Δ_{1,i_1,i_2} in the case $r = 3$, $l = 1$ of Theorem 2.3 by the sum of the two "3-cycles" $b_{12}b_{23}b_{31}$, $b_{13}b_{32}b_{21}$, one of which is, however, zero because of Theorem 3.2. The fact that one is zero is generalized in the next theorem.

THEOREM 3.3. *We assume A, B as in Theorem* 3.2. *Assume further that for the matrix B all s-cycles $b_{i_1i_2}b_{i_2i_3} \cdots b_{i_{s-1}i_s}b_{i_si_1} = 0$, $s \leq r-1$, $i_\alpha \neq i_\beta$. Then at most one r-cycle can be nonzero among the cycles involving a fixed set of indices.*

Proof. There is no loss in generality in assuming that

$$b_{12}b_{23} \cdots b_{r-1,r}b_{r1} \neq 0.$$

This implies $b_{21} = b_{32} = \cdots b_{r-1,r} = b_{1r} = 0$ by 3.2. We now show that all $b_{ik} = 0$, $i \neq k$, $i, k = 1, \dots, r$, unless $k = i + 1$, or $i = r$, $k = 1$. There are two cases:

(A). $i > k$: if $b_{ik} \neq 0$, then the following cycle is also $\neq 0$:

$$b_{ik} b_{k,k+1} \cdots b_{i-1,i} .$$

This cycle involves $i - 1 - k + 1 + 1 = i - k + 1$ factors; since $i = r$, $k = 1$ is excluded, $i - k < r - 2$; hence the cycles needed are at most of length $r - 1$.

(B). $i < k \neq i + 1$: here we use the cycle

$$b_{ik} b_{k,k+1} \cdots b_{r-1,r} b_{r1} b_{12} \cdots b_{i-1,i} .$$

This cycle is of length $r - k + 1 + i - 1 + 1 = r - (k - i) + 1 < r$, since $k - i > 1$.

THEOREM 3.4. *Let A, B be as in Theorem 3.2. If all lower dimensional cycles in B are zero then all $(n - 1)$-cycles are zero.*

Proof. This follows from the relations (2_r) for $r = n - 1$ and Theorem 3.3 for $r = n - 1$. Use further the L-property for A, $A^r B$. This gives a set of equations

$$\sum (\alpha_\ell \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{n-2}})^r (\alpha_\ell - \alpha_i)\, \Delta_{\ell, i_1, \dots, i_{n-2}} = 0$$

where in Δ we only consider the sum of the $n - 1$ cycles, only one of which can be different from zero. The product $\alpha_\ell \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{n-2}}$ containing $n - 1$ factors α is equal to $(\alpha_1 \alpha_2 \cdots \alpha_n)/\alpha_j$ with $j \neq \ell$. Hence, we may again employ the van der Monde treatment of the proof of Theorem 3.2, this time for the $1/\alpha_j$'s instead of the α_j's to show that all $\Delta_{\ell, i_1, \dots, i_{n-2}} = 0$ for any fixed ℓ.

THEOREM 3.5. *Let A, B be as in Theorem 3.2. If all lower dimensional cycles in B are zero then all n-cycles are zero.*

Proof. This follows immediately from Theorem 3.3.

4. SPECIAL CASES

Here we add to the conditions assumed in Theorem 3.2: the following pairs have the L-property

$$A, \quad B^2$$
$$AB, \quad BA.$$

We then obtain the following theorems.

THEOREM 4.1. *For $n \leqslant 6$ the commutator $AB - BA$ is nilpotent.*

Remark. It can happen to be more suitable to assume the L-property for the pair B, $A^{-1}BA$ than for the pair AB, BA. E.g., if B has all its characteristic roots equal then the pairing of the characteristic roots for B, $A^{-1}BA$ is no problem: $B - A^{-1}BA$ is then a nilpotent matrix and $B^{-1}A^{-1}BA$ is unipotent. Alternatively, if AB has all its characteristic roots equal, then $AB - BA$ is clearly nilpotent under our L-property assumptions, but the pairing of the characteristic roots for B, $A^{-1}BA$ is not obvious.

Proof of 4.1. The commutator $AB - BA$ under our assumptions on A, B is the matrix

$$((\alpha_i - \alpha_j)\, b_{ij}).$$

For this matrix to be nilpotent it is clearly sufficient to show that all principal minors are zero and for this it is sufficient to show that all r-cycles $b_{i_1 i_2} \cdots b_{i_r i_1} = 0$, $r = 2, \ldots, n$. This is true for $n \leqslant 6$:

$n = 2$: obvious from Theorem 3.2.

$n = 3$: this follows from Theorems 3.2, 3.5.

$n = 4$: this follows from Theorems 3.2, 3.4, 3.5.

$n = 5$: this follows from $r = 2, 4, 5$ from Theorems 3.2, 3.4, 3.5 after the case $r = 3$ is established:

Denote the sum of the two 3-cycles involving the indices i, k, j by E_{ikj}. As pointed out earlier, one of them vanishes by Theorem 3.2. Then keep i fixed, e.g., $i = 1$, but vary k, j over all possible values $\neq 1$. Then use relations (2_r) for $r = 3$, $\ell = 1$:

$$\sum_{j,k \neq 1} C_{1,k,j} E_{1kj} = 0.$$

We next replace the pair A, B by A, $A^r B$ as in the proof of Theorem 3.2 and obtain the system

$$\sum C_{1,k,j} (\alpha_1 \alpha_k \alpha_j)^r E_{ikj} = 0.$$

If all $\alpha_k \alpha_j$, $k, j \neq 1$, are different then we can apply the "van der Monde treatment" used earlier to show that all $E_{1kj} = 0$. If however, e.g.,

$$\alpha_2 \alpha_3 = \alpha_4 \alpha_5 \tag{3}$$

then we can conclude that

$$C_{1,2,3} E_{123} + C_{1,4,5} E_{145} = 0.$$

We then turn to the E_{2kj}, $k, j \neq 2$. Since $E_{213} = E_{123}$ is contained among them we must have

$$\alpha_1 \alpha_3 = \alpha_4 \alpha_5 \tag{4}$$

or $E_{123} = 0$. However, the relations (3), (4) are in contradiction with $\alpha_1 \neq \alpha_2$.

$n = 6$: as in the previous cases the r-cycles, $r = 2, 5, 6$, will vanish as a consequence of the theorems proved under 2. We now study $r = 3$. As in the case $n = 5$ we are prevented from using the "van der Monde treatment" for the systems of equation,

$$\sum_{k,j \neq 1} C_{1,k,j} E_{1kj} = 0,$$

and the resulting equations corresponding to the pair A, $A^r B$, $r = 1, 2, \ldots$, if two products $\alpha_k \alpha_j$, $k \neq j$, $k, j \neq 1$, coincide. We may take the special case

$$\alpha_2 \alpha_3 = \alpha_4 \alpha_5 . \tag{5}$$

Note that no other $\alpha_k \alpha_j$, $k, j \neq 1$, can coincide with these two products, in virtue of $\alpha_i \neq \alpha_k$. If then $E_{123} \neq 0$, $E_{145} \neq 0$ we proceed as in the case $n = 5$ to the system

$$\sum C_{2,k,j} E_{2kj} = 0$$

and conclude that

$$\alpha_2 \alpha_1 \alpha_3 = \alpha_2 \alpha_k \alpha_j$$

for some $k, j \neq 1, 2, 3$. Hence $\alpha_k \alpha_j = \alpha_4 \alpha_6$ or $\alpha_5 \alpha_6$ since $\alpha_4 \alpha_5$ is excluded by $\alpha_1 \alpha_2 \alpha_3 = \alpha_1 \alpha_4 \alpha_5$. We may assume, e.g.,

$$\alpha_2 \alpha_1 \alpha_3 = \alpha_2 \alpha_4 \alpha_6 \tag{6}$$

without loss of generality.

We now turn to the system

$$\sum C_{3,k,j} E_{3kj} = 0$$

which contains the nonvanishing term $C_{312} E_{312}$; hence

$$\alpha_3 \alpha_1 \alpha_2 = \alpha_3 \alpha_k \alpha_j$$

for some elements $k, j \neq 1, 2, 3$, and, because of the choices made earlier, we are compelled to take

$$\alpha_3 \alpha_1 \alpha_2 = \alpha_3 \alpha_5 \alpha_6 . \tag{7}$$

On the other hand, equation $\alpha_1 \alpha_2 \alpha_3 = \alpha_1 \alpha_4 \alpha_5$ can be used to obtain relations for the products $\alpha_4 \alpha_k \alpha_\ell$ and $\alpha_5 \alpha_k \alpha_\ell$ while $\alpha_2 \alpha_1 \alpha_3 = \alpha_2 \alpha_4 \alpha_6$ can be used to obtain relations for the products $\alpha_6 \alpha_k \alpha_\ell$. The relations for the $\alpha_4 \alpha_k \alpha_\ell$ can be chosen in two different ways, either

$$\alpha_4 \alpha_1 \alpha_5 = \alpha_4 \alpha_2 \alpha_6 \quad \text{or} \quad \alpha_4 \alpha_3 \alpha_6 .$$

If we take the first one then

$$\alpha_5\alpha_1\alpha_4 = \alpha_5\alpha_3\alpha_6 \tag{8}$$

follows. Further

$$\alpha_6\alpha_2\alpha_4 = \alpha_6\alpha_3\alpha_5 \tag{9}$$

follows as a consequence. Altogether the following set of relations emerge (the alternative choices lead to analogous sets):

$$\begin{aligned} &\alpha_2\alpha_3 = \alpha_4\alpha_5\,, \qquad \alpha_1\alpha_3 = \alpha_4\alpha_6\,, \qquad \alpha_1\alpha_2 = \alpha_5\alpha_6\,, \\ &\alpha_1\alpha_5 = \alpha_2\alpha_6\,, \qquad \alpha_1\alpha_4 = \alpha_3\alpha_6\,, \qquad \alpha_2\alpha_4 = \alpha_3\alpha_5\,. \end{aligned} \tag{10}$$

They lead to no contradiction with $\alpha_i \neq \alpha_k$ unless the characteristic of the field is 2, for the set

$$\begin{aligned} &\alpha_1 = -\alpha_6 = a, \qquad a \neq b \neq c \neq a, \\ &\alpha_2 = -\alpha_5 = b, \\ &\alpha_3 = -\alpha_4 = c, \end{aligned} \tag{11}$$

satisfies the above relations and is the only set satisfying them, since $\alpha_1{}^2 = \alpha_6{}^2$, $\alpha_2{}^2 = \alpha_5{}^2$, $\alpha_3{}^2 = \alpha_4{}^2$ follows from the above relations. The relation $\alpha_2\alpha_4 = \alpha_3\alpha_5$ implies that another pair of E_{1jk}'s, namely E_{124}, E_{135} may not vanish.

The only remaining pair $\alpha_{i_1}\alpha_{i_2} = \alpha_{k_1}\alpha_{k_2}$, with i's and k's $\neq 1$, would be

$$\alpha_2\alpha_5 = \alpha_3\alpha_4\,.$$

However, it too does not come in question. For $E_{125} \neq 0$ would imply that

$$\alpha_1\alpha_5 = \alpha_k\alpha_\ell$$

with $k, \ell \neq 2$, but we already know that

$$\alpha_1\alpha_5 = \alpha_2\alpha_6$$

and the only product with suffixes $\neq 2, 1, 5, 6$ is $\alpha_3\alpha_4$ which is equal to $\alpha_2\alpha_5$.

Hence, under our choices the only possibly nonvanishing pairs E_{1jk} are

$$E_{123}\,,\ E_{145}\,;\ E_{124}\,,\ E_{135}\,.$$

A similar treatment holds for the other E_{ijk}, showing that the following sets of two pairs are the only ones which will lead to nonvanishing E's under the assumption that $E_{123} \neq 0$:

$$\begin{array}{ll} i = 1 & E_{123}\,, E_{145}\,;\ E_{124}\,, E_{135} \\ i = 2 & E_{123}\,, E_{246}\,;\ E_{124}\,, E_{236} \\ i = 3 & E_{123}\,, E_{356}\,;\ E_{135}\,, E_{236} \\ i = 4 & E_{145}\,, E_{246}\,;\ E_{124}\,, E_{456} \\ i = 5 & E_{145}\,, E_{356}\,;\ E_{135}\,, E_{456} \\ i = 6 & E_{236}\,, E_{456}\,;\ E_{246}\,, E_{356}\,. \end{array} \tag{12}$$

If any of these E's were zero, then all would be. For, let, e.g., $E_{124} = 0$; then both $b_{12}b_{24}b_{41}$ and $b_{21}b_{14}b_{42} = 0$. This implies that these 3-cycles involve an element b_{ik} such that $b_{ik} = b_{ki} = 0$. However, b_{12} in involved in E_{123}, b_{14} in E_{145} and b_{24} in E_{246} which are all assumed nonzero.

We next go from the E_{ikj} to the actual 3-cycles $b_{ik}b_{kj}b_{ji}$ or $b_{ki}b_{ij}b_{jk}$ one of which must vanish. Let us choose $b_{12}b_{23}b_{31} \neq 0$ so that $b_{21} = b_{32} = b_{13} = 0$. It can then be shown that this fixes the choice for all other nonvanishing pairs of 3-cycles so that:

$$\begin{aligned} b_{21}b_{14}b_{42} &= b_{14}b_{45}b_{51} = b_{13}b_{35}b_{51} = b_{42}b_{26}b_{64} \\ &= b_{32}b_{26}b_{63} = b_{63}b_{35}b_{56} = b_{64}b_{45}b_{56} = 0. \end{aligned} \tag{13}$$

Under the choices made the matrix B is then

$$B = \begin{bmatrix} \beta_1 & b_{12} & 0 & 0 & b_{15} & b_{16} \\ 0 & \beta_2 & b_{23} & b_{24} & b_{25} & 0 \\ b_{31} & 0 & \beta_3 & b_{34} & 0 & b_{36} \\ b_{41} & 0 & b_{43} & \beta_4 & 0 & b_{46} \\ 0 & b_{52} & b_{53} & b_{54} & \beta_5 & 0 \\ b_{61} & b_{62} & 0 & 0 & b_{65} & \beta_6 \end{bmatrix}.$$

Since, by assumption A, B^2 are a pair with property L, only the same E_{ikj} are possibly $\neq 0$ for B^2 are as for B. We have therefore to study the elements $b_{ik}^{(2)}$ of B^2 and distinguish the following two cases:

$$\begin{array}{llll} \text{I.} & b_{ik}^{(2)}b_{kj}^{(2)}b_{ji}^{(2)} = 0 & \text{if} & b_{ik}b_{kj}b_{ji} = 0; \\ \text{II.} & b_{ki}^{(2)}b_{ij}^{(2)}b_{jk}^{(2)} = 0 & \text{if} & b_{ik}b_{kj}b_{ji} = 0. \end{array}$$

As for B, the nonvanishing of any of the pertinent 3-cycles determines the possible nonvanishing 3-cycles in all other E_{ikj}'s.

I. Here we have the equation

$$b_{21}^{(2)} = b_{23}b_{31} + b_{24}b_{41} = 0.$$

After multiplication with the nonvanishing b_{12} this gives

$$E_{123} + E_{124} = 0.$$

But to this equation the van der Monde treatment of 3.2 can be applied by using the L-property for A, AB. Since $\alpha_3 \neq \alpha_4$ this gives $E_{123} = E_{124} = 0$, and, hence, all the E's vanish.

II. Here we begin by observing that in the pairs

$$b_{16}\,,\, b_{61}\,;\; b_{25}\,,\, b_{52}\,;\; b_{34}\,,\, b_{43} \tag{14}$$

one element must vanish. Further, each of the vanishing elements $b_{ik}^{(2)}$ is of a form analogous to

$$b_{12}^{(2)} = (\beta_1 + \beta_2)\, b_{12} + b_{15}b_{52} + b_{16}b_{62} = 0. \tag{15}$$

Here the last two terms contain two elements from (14) from different pairs. So do all the other relevant $b_{ik}^{(2)}$. If both of these are zero and this does happen under any combination of the possible values in (14), then we can conclude that the corresponding $\beta_i + \beta_k = 0$. We then replace the pair A, B by the pair A, AB and note that AB does have zero entries at exactly the same places as B. Hence, the matrix $(AB)^2$ will either come under Case I or it will lead to the relation

$$\alpha_i\beta_i + \alpha_k\beta_k = 0.$$

This combined with $\beta_i + \beta_k = 0$ implies $\alpha_i = \alpha_k$ which is a contradiction.

We are left with the study of the 4-cycles. They are in a dual situation to the 3-cycles. First of all, because of $\binom{5}{3} = \binom{5}{2}$, there are as many 4-cycles $b_{1i_2}b_{i_2i_3}b_{i_3i_4}b_{i_41}$ as there are 3-cycles $b_{1i_2}b_{i_2i_3}b_{i_31}$. Further, when applying the "van der Monde treatment" of 3.2 to (3_4), $n = 6$, $\ell = 1$ we have to watch out for coinciding products $\alpha_1\alpha_k\alpha_\ell\alpha_t$. But this product is equal to $\prod \alpha_i/\alpha_r\alpha_s$, $r, s \neq 1, k, \ell, t$. Hence we are again concerned only with products of two α's and these have been fixed under (10). Thus for $\ell = 1$, only the four following combinations come in question:

$$1, 4, 5, 6;\; 1, 2, 3, 6 \qquad \text{and} \qquad 1, 3, 5, 6;\; 1, 2, 4, 6.$$

From the vanishing of all 3-cycles it follows that in each 3-cycle $b_{ik}b_{kj}b_{ki}$ at least one factor and its "transpose" vanish simultaneously. It follows by inspection that all the above 4-cycles must vanish.

THEOREM 4.2. *Let A, B be two 3×3 matrices, with property L and characteristic roots α_i paired with β_i. It is further assumed that the following pairs have the L-property:*

$$AB, \quad BA$$
$$A, \quad B^2.$$

It then follows that $AB - BA$ is nilpotent.

Proof. The case where $\alpha_i \neq \alpha_k$, $i \neq k$ is already covered by Theorem 4.1. We may assume that neither A nor B have a characteristic root of multiplicity 3. For, otherwise the theorem would be trivial. We thus assume that A has at most a double root and that it is in Jordan canonical form. We have then the cases:

I. $$A = \begin{bmatrix} \alpha_1 & & \\ & \alpha_1 & \\ & & \alpha_3 \end{bmatrix} \quad \alpha_1 \neq \alpha_3 .$$

II. $$A = \begin{bmatrix} \alpha_1 & 1 & \\ & \alpha_1 & \\ & & \alpha_3 \end{bmatrix} \quad \alpha_1 \neq \alpha_3 .$$

I. By 2.1 we know that in the case $\alpha_1 = \alpha_2$

$$b_{23}b_{32} + b_{13}b_{31} = 0.$$

The commutator $AB - BA$ is hence equal to

$$\begin{bmatrix} 0 & 0 & (\alpha_1 - \alpha_3)\, b_{13} \\ 0 & 0 & (\alpha_1 - \alpha_3)\, b_{23} \\ (\alpha_3 - \alpha_1)\, b_{31} & (\alpha_3 - \alpha_1)\, b_{32} & 0 \end{bmatrix}.$$

The trace and the determinant of this matrix are clearly zero. The sum of the 2-dimensional minors is also zero because of the above relation and $\alpha_1 = \alpha_2$. Hence $AB - BA$ is nilpotent.

II. We use the fact that

$$\lambda A + B = \begin{bmatrix} \lambda\alpha_1 + b_{11} & \lambda + b_{12} & b_{13} \\ b_{21} & \lambda\alpha_1 + b_{22} & b_{23} \\ b_{31} & b_{32} & \lambda\alpha_3 + b_{33} \end{bmatrix}$$

has the characteristic root $\lambda\alpha_1 + \beta_1$ for all values of λ. Hence the following determinant vanishes for all λ:

$$\begin{vmatrix} b_{11} - \beta_1 & \lambda + b_{12} & b_{13} \\ b_{21} & b_{22} - \beta_1 & b_{23} \\ b_{31} & b_{32} & \lambda(\alpha_3 - \alpha_1) + b_{33} - \beta_1 \end{vmatrix}$$

Hence the coefficients of λ and of λ^2 vanish. The coefficient of λ^2 is: $(\alpha_3 - \alpha_1)\, b_{21}$, hence

$$b_{21} = 0. \tag{16}$$

The coefficient of λ is:

$$(\alpha_3 - \alpha_1) \begin{vmatrix} b_{11} - \beta_1 & b_{12} \\ 0 & b_{22} - \beta_1 \end{vmatrix} + b_{23}b_{31} = 0.$$

The same equation holds for β_2, but $b_{33} = \beta_3$. We next show that

$$b_{23}b_{31} = 0. \tag{17}$$

For this we use the fact that A, B^2 are a pair with the L-property. Again, the element in the (2,1) position of B^2 is 0, but this is $b_{23}b_{31}$. Hence β_1, β_2 are the characteristic roots of $\left(\begin{smallmatrix} b_{11} & b_{22} \\ 0 & b_{22} \end{smallmatrix}\right)$ so that $\beta_1 = b_{11}$, $\beta_2 = b_{22}$.

We next show

$$b_{13}b_{31} + b_{23}b_{32} = 0. \tag{18}$$

This follows from the fact that the two following polynomials must coincide:

$$\begin{vmatrix} b_{11} - x & b_{12} \\ 0 & b_{22} - x \end{vmatrix} (b_{33} - x) = \begin{vmatrix} b_{11} - x & b_{12} & b_{13} \\ 0 & b_{22} - x & b_{23} \\ b_{31} & b_{32} & b_{33} - x \end{vmatrix}.$$

Comparing the coefficients of x gives (18).
Finally compute

$$AB - BA = \begin{bmatrix} 0 & b_{22} - b_{11} & (\alpha_1 - \alpha_3)\, b_{13} + b_{23} \\ 0 & 0 & (\alpha_1 - \alpha_3)\, b_{23} \\ (\alpha_3 - \alpha_1)\, b_{31} & -b_{31} + (\alpha_3 - \alpha_1)\, b_{32} & 0 \end{bmatrix}.$$

This matrix has trace 0, the sum of its 2-dimensional principal minors is 0 because of (16), (17), (18). The determinant is also 0.

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